Due Today

3.2 – Norm, Dot Product, and Distance in R^n

Definition 1: If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the **norm** of $\mathbf{v}$ (also called its **length** or **magnitude**) is denoted by $||\mathbf{v}||$ [by this author], and is defined by the formula

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Theorem 3.2.1 Properties of the norm of a vector

If $\mathbf{v}$ is a vector in R^n and k is any scalar, then:

a) $||\mathbf{v}|| \geq 0$
b) $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$
c) $||k\mathbf{v}|| = |k|||\mathbf{v}||$

good to prove

The norm of a **unit vector** is 1.

$$\left\|\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\right\| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1$$

We can obtain a unit vector from a nonzero vector $\mathbf{v}$ by multiplying by the reciprocal of its length. This process is called **normalizing** the vector.

1. Find the norm of $\mathbf{v}$, and a unit vector that is oppositely directed to $\mathbf{v}$.

a. $\mathbf{v} = (2, 2, 2)$
b. $\mathbf{v} = (1, 0, 2, 1, 3)$

$$b.\ ||\vec{v}|| = \sqrt{1^2 + 2^2 + 1^2 + 3^2} = \sqrt{15}$$

$$-\frac{1}{||\vec{v}||}\vec{v} = \left(-\frac{1}{\sqrt{15}}, 0, -\frac{2}{\sqrt{15}}, -\frac{1}{\sqrt{15}}, -\frac{3}{\sqrt{15}}\right)$$

$(2,-3,4) = 2\vec{e}_1 - 3\vec{e}_2 + 4e_3 = 2(1,0,0) - 3(0,1,0) + 4(0,0,1)$

The **standard unit vectors in R^n** are the standard basis vectors for R^n, $\mathbf{e}_1 = (1,0,0,\ldots,0), \mathbf{e}_2 = (0,1,0,\ldots,0), \ldots, \mathbf{e}_n = (0,0,0,\ldots,1)$.

Every vector in R^n can be expressed as a linear combination of the standard basis vectors:

$$\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n + \vec{e}_n$$

Definition 2: If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then we denote the **distance** between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u},\mathbf{v})$ and define it to be

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + \cdots + (u_n-v_n)^2}$$

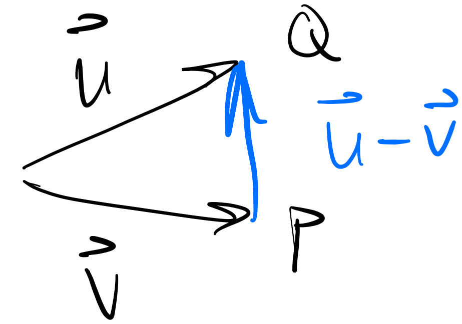
4. Evaluate the given expression with $\mathbf{u} = (2,-2,3)$, $\mathbf{v} = (1,-3,4)$, and $\mathbf{w} = (3,6,-4)$.

a. $\|\mathbf{u}+\mathbf{v}+\mathbf{w}\|$

b. $\|\mathbf{u}-\mathbf{v}\| \leftarrow$ distance

c. $\|3\mathbf{v}\| - 3\|\mathbf{v}\| = 0$

d. $\|\mathbf{u}\| - \|\mathbf{v}\|$



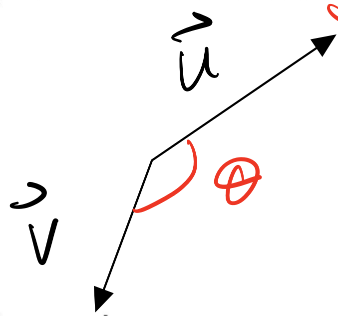
b. $\vec{u} - \vec{v} = (2-1, -2+3, 3-4) = (1, 1, -1)$

$\|\vec{u}-\vec{v}\| = \sqrt{3}$

d. $\|\vec{u}\| = \sqrt{4+4+9} = \sqrt{17}$, $\|\vec{v}\| = \sqrt{1+9+16} = \sqrt{26}$

$\|\vec{u}\| - \|\vec{v}\| = \sqrt{17} - \sqrt{26}$

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in R^n positioned so that their initial points coincide. The **angle between $\mathbf{u}$ and $\mathbf{v}$** is the angle θ determined by $\mathbf{u}$ and $\mathbf{v}$ such that $0 \leq \theta \leq \pi$.

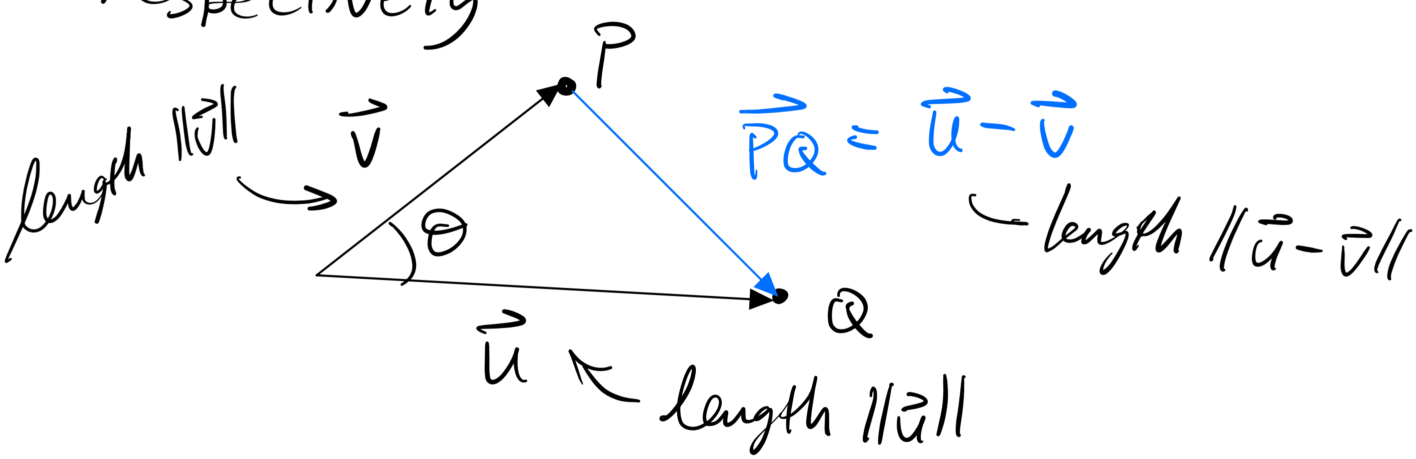


Definition 3: If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the **dot product** or **Euclidean inner product** of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

$\vec{u}\vec{v}$ is not a thing

Definition 4: If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are vectors in R^n, then the **dot product** or **Euclidean inner product** of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by
$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$.

To connect these two definitions, let $\vec{u}$ & $\vec{v}$ share an initial point in R^3 and let Q & P be their terminal points, respectively



By the Law of Cosines,

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\Rightarrow \|\vec{u}\|\|\vec{v}\|\cos\theta = \frac{1}{2}\left(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2\right)$$

$$= \frac{1}{2}\Big(u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - (u_1-v_1)^2 - (u_2-v_2^2) - (u_3-v_3)^2\Big)$$

$$\boxed{\|\vec{u}\|\|\vec{v}\|\cos\theta = u_1v_1 + u_2v_2 + u_3v_3 = \vec{u}\cdot\vec{v}}$$

In general, $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$

10. Find $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\cdot\mathbf{u}$, and $\mathbf{v}\cdot\mathbf{v}$.

a. $\mathbf{u} = (1, 1, -2, 3)$, $\mathbf{v} = (-1, 0, 5, 1)$

b. $\mathbf{u} = (2, -1, 1, 0, -2)$, $\mathbf{v} = (1, 2, 2, 2, 1)$

b. $\vec{u}\cdot\vec{v} = 2(1) + (-1)(2) + 1(2) + 0(2) + (-2)(1)$

$= 0 = \|\vec{u}\|\|\vec{v}\|\cos\theta \Rightarrow \theta = \frac{\pi}{2}$

$\vec{u}\cdot\vec{u} = 2^2 + (-1)^2 + 1^2 + 0^2 + (-2)^2 = 10 \leftarrow \|\vec{u}\|^2$

$\vec{v}\cdot\vec{v} = 1^2 + 2^2 + 2^2 + 2^2 + 1^2 = 14 \leftarrow \|\vec{v}\|^2$

11. Find the Euclidean distance between $\mathbf{u}$ and $\mathbf{v}$ and the cosine of the angle between those vectors. State whether that angle is acute, obtuse, or $90°$.

a. $\mathbf{u} = (3, 3, 3)$, $\mathbf{v} = (1, 0, 4)$

b. $\mathbf{u} = (0, -2, -1, 1)$, $\mathbf{v} = (-3, 2, 4, 4)$

a) $d(\vec{u}, \vec{v}) = \sqrt{(3-1)^2 + (3-0)^2 + (3-4)^2} = \sqrt{14}$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{15}{3\sqrt{3}\sqrt{17}} = \frac{15}{3\sqrt{51}}$$

Since $\cos\theta > 0$, $0 < \theta < \pi/2$ (acute).

b) $\vec{u} \cdot \vec{v} < 0 \Rightarrow \theta$ is obtuse.

Theorem 3.2.2 Properties of the dot product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and if k is a scalar, then:

a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (symmetry property)

b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributive property)

c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ (homogeneity property)

d) $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (positivity property)

Pf: b) Let $\vec{u} = (u_1, u_2, \ldots, u_n)$, $\vec{v} = (v_1, v_2, \ldots, v_n)$, $\vec{w} = (w_1, w_2, \ldots, w_n)$

Then $\vec{u} \cdot (\vec{v} + \vec{w}) = (u_1, u_2, \ldots, u_n) \cdot (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$

(def of vector addition)

$= u_1(v_1 + w_1) + u_2(v_2 + w_2) + \cdots + u_n(v_n + w_n)$

(def of dot product)

$= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \cdots + u_nv_n + u_nw_n$

(dist. prop of mult. of real #s)

$= u_1v_1 + u_2v_2 + \cdots + u_nv_n + u_1w_1 + u_2w_2 + \cdots + u_nw_n$

(comm. prop. of add. of real #s)

$= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

(def of dot product) ✓

Theorem 3.2.3 More properties of the dot product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and if k is a scalar, then:

a) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

d) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

e) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

— a necessary result for the def of dot product

Theorem 3.2.4 Cauchy-Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \dots u_n)$ and $\mathbf{v} = (v_1, v_2, \dots v_n)$ are vectors in R^n, then $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$ or in terms of components

$$|u_1 v_1 + u_2 v_2 + \cdots + u_n v_n| \leq (u_1^2 + u_2^2 + \cdots u_n^2)^{1/2}(v_1^2 + v_2^2 + \cdots v_n^2)^{1/2}$$

Theorem 3.2.5 Triangle Inequalities

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n, then:

a) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (triangle inequality for vectors)

b) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (triangle inequality for distances)

Pf: a) $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$

$\leq \|\vec{u}\|^2 + 2|\vec{u}\cdot\vec{v}| + \|\vec{v}\|^2$

(prop. of abs. value)

$\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$

(Cauchy-Schwarz inequality)

$= (\|\vec{u}\| + \|\vec{v}\|)^2$

Taking square roots yields $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$.

b) follows from (a) and Def. 2.

Note: An expression like $\vec{u}\cdot\vec{v}\cdot\vec{w}$ has no meaning. But we can consider matrix mult. using the dot product:

If A is an $n\times n$ matrix and $\vec{u}$ & $\vec{v}$ are $n\times 1$ matrices (columns), then

$$\underset{n\times 1}{\underset{n\times n}{A}\underset{n\times 1}{\vec{u}}}\cdot\underset{n\times 1}{\vec{v}} = \underset{1\times n}{\vec{v}^T}\,\underset{n\times 1}{(A\vec{u})} = \underset{1\times n}{(\underset{1\times n}{\vec{v}^T}\underset{n\times n}{A})}\,\underset{n\times 1}{\vec{u}} = \underset{1\times n}{\underset{n\times 1}{(\underset{n\times n}{A^T}\underset{n\times 1}{v})}^T}\,\underset{n\times 1}{\vec{u}} = \vec{u}\cdot A^T\vec{v}$$

So $A\vec{u}\cdot\vec{v} = \vec{u}\cdot A^T\vec{v}$. Likewise,

$$\vec{u}\cdot A\vec{v} = A^T u\cdot\vec{v}.$$

Last note: If we represent A $(m\times n)$ and B $(n\times r)$ as $A = \begin{bmatrix}\vec{r}_1\\ \vec{r}_2\\ \vdots\\ \vec{r}_m\end{bmatrix}$ and $B = \begin{bmatrix}\vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_r\end{bmatrix}$,

then

$$AB = \begin{bmatrix}\vec{r}_1\cdot\vec{c}_1 & \vec{r}_1\cdot\vec{c}_2 & \cdots & \vec{r}_1\cdot\vec{c}_r\\ \vdots & \vdots & & \vdots\\ \vec{r}_m\cdot\vec{c}_1 & \vec{r}_m\cdot\vec{c}_2 & \cdots & \vec{r}_m\cdot\vec{c}_r\end{bmatrix}$$